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Purely Lifting Module.

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Abstract

In this paper we will generalize the concept of a lifting module. Various properties and characterizations of a general case; the pure lifting module, are established and some other known results has been generalized.

Keywords: Pure and small sub-modules, lifting, regular, hollow, multiplication, prime, faithful, and cancellation modules.

Introduction

A submodule A of a module M is called small if whenever A+B=M, for some sub-module B of M, then B=M, and a module M is called a lifting module if for every sub-module N of M, there exists a direct summand A of M such that $A \subseteq N$ and N∩B small in M, where M = A ⊕ B, for some sub-module B of M, see [1]. Now recall that a sub-module N of an R-module M is called a pure sub-module if IM∩N=IN, for every finitely generated ideal I of R, see [2], and [3]. One can easily show that every direct summand is pure.

These observations lead us to introduce the following concept an R-module M is called a purely lifting module if for every sub-module N of M , there exists a pure sub-module K of M such that K ⊆ N and N/K is small in M/K. This work consists of three sections. In section one we give a review of some of the properties of lifting module, see [1]. Also we add some new results, for example we prove that for a finitely generated, faithful and multiplication R-module. M is lifting if and only if R is lifting, see (1.6). In section two, we give the definition of the purely lifting modules with some examples and basic properties. And we give a characterization for a purely lifting module, see (2.9). We prove that, for a V-module M, M is purely lifting if and only if M is a regular module, see (2.4). In section 3, we note that all rings considered in this paper are unitary (left R- modules).

§1 Lifting modules

In this section we recall some basic results of lifting modules. We also add some new results. We start by the following proposition, see [1].

Proposition 1.1: Let M be an R –module. The following statements are equivalents
(1) M is lifting.

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(2) Every sub-module N of M can be written as N = A ⊕ S, where A is a direct summand of M and S is small in M.

(3) For every sub-module N of M, there exists a direct summand K of M such that K ⊆ N and N/K is small in M/K.

Theorem 1.2: [1] Let R be a ring with J(R) =0. Then the following are equivalent

(1) R is semisimple.

(2) Every R module is lifting.

Let R be an integral domain, recall that an R-module M is a prime R-module if ann(a)=ann (M), for all 0≠a∈ M, [4]. We know that the sum of all simple submodules of M is called the socle of M and is denoted by Soc (M), moreover if Soc (M) =0 then M has no simple submodule, and if Soc (M) =M then M is semisimple module, see [5].

Proposition 1.3: Let R be an integral domain and let M be a prime R-module such that Soc (M)≠0, then M is semisimple.

Proof: Since Soc (M) ≠ 0 and Soc (M) =∑ Rx, where Rx is a simple sub-module of M, then there exists x_0 ∈ M such that Rx_0 is simple. Now consider the following short exact sequence

$$0 \longrightarrow \mathrm{ann}(x_0) \xrightarrow{i} R \xrightarrow{f} Rx_0 \rightarrow 0,$$

Where i is the inclusion map and f(r) = rx_0 for all r in R. By the first isomorphism theorem $Rx_0 \cong (R/\ker f) = (R/\mathrm{ann}(x_0))$. Now let Rx be any cyclic sub-module of M such that x ≠ 0. Since M is a prime module, then ann(x) = ann(x_0). Now consider the short exact sequence

$$0 \longrightarrow \mathrm{ann}(x) \xrightarrow{i1} R \xrightarrow{g} Rx \longrightarrow 0,$$

Where i1 is the inclusion map and g(r)=rx, for all r in R. By the first isomorphism theorem $Rx \cong \left(\frac{R}{\mathrm{ann}g}\right) = \left(\frac{R}{\mathrm{ann}(x)}\right)$, and hence $Rx \cong \left(\frac{R}{\mathrm{ann}x}\right) = \left(\frac{R}{\mathrm{ann}(x_0)}\right) \cong Rx_0$, but Rx_0 is simple therefore Rx is simple. Thus M=Soc (M) and hence M is semisimple.

Corollary 1.4: Let R be an integral domain and let M be a prime R-module such that Soc (M) ≠ 0, then M is a lifting module.

Proof: Clear.

Proposition 1.5: Let M be a projective R-module. If M is a lifting module then M/N has projective cover, for each sub-module N of M.

Proof: Let N be a sub-module of M. Since M is a lifting module, then M = A ⊕ B, where A⊆ N and B∩N is small in B. Now consider the following two short exact sequences

$$0 \longrightarrow N \xrightarrow{i1} N+B \xrightarrow{\pi 1} \frac{N+B}{N} \longrightarrow 0$$

$$0 \longrightarrow N\cap B \xrightarrow{i2} B \xrightarrow{\pi 2} \frac{B}{N\cap B} \longrightarrow 0$$

Where i1 and i2 are inclusion maps and $\pi 1$ and $\pi 2$ are the natural epimorphisms. By the second isomorphism theorem

$(M/N)=((N+B)/N)\cong(B/(N\cap B))$, since M is projective and B is a summand of M, then by [5] B is projective. But $N\cap B$ is small in B, therefore B is projective cover of $\frac{B}{N\cap B}$, and hence B is projective cover of $\frac{M}{N}$.

Recall that an R-module M is called a multiplication module if N = (N: M) M, for every sub-module N of M, see [6].

Proposition 1.6: Let M be a finitely generated, faithful and multiplication R-module. M is lifting if and only if R is lifting.

Proof →) Let I be an ideal of R. Since M is multiplication, then IM= (IM:M) M. Since M is a lifting module, then by (1.1) IM =A ⊕B, where A is a summand of M and B is small in M. But M is a multiplication module, then $A = (A:M)M$, and $B = (B:M)M$. Now:

IM = (A:M)M ⊕ (B:M)M

=((A:M)+(B:M))M, and,0 = (A:M)M ∩ (B:M)M = ((A:M) ∩ (B:M))M , thus (A:M) ∩ (B:M) ⊆ annM = 0 and hence IM = ((A:M) ⊕ (B:M))M. Since M is finitely generated, faithful and multiplication R-module, then by [7], M is a cancellation module and hence I = (A: M) ⊕ (B: M). we claim that (A: M) is a summand of R, to show that. Since A is a summand of M, then M =A ⊕ A1, for some sub-module A1 of M. Now M = RM = A ⊕ A1 = (A:M)M ⊕ (A1:M)M = ((A:M) ⊕ (A1:M))M. But M is cancellation, so R = (A: M) ⊕ (A1: M). to show that (B: M) is small in R. Let (B: M) +J= R, for some ideal J of R.
So B+JM= (B: M) M+JM= ((B: M) +J) M= M, but B is small in M, therefore JM = M = RM. Since M is a cancellation module, then J = R. By (1.1) R is a lifting module.

←) By the same way, one can prove the converse.

Proposition 1.7: Let M be an R- module and let $\bar{R} = \frac{R}{annM}$ then M is lifting as R-module if and only if M is lifting as $\bar{R}$-module.

Proof: Clear.

Recall that a sub-module N of an R – module M is called fully invariant if for every endomorphism f: M → M, f (N) ⊆ N, see [8] , and an R- module M is called fully stable if every sub- module of M is fully invariant, see [8].

Proposition 1.8: Let $M=\bigoplus_{i=1}^{n} M_i$ be a fully stable R- module. The module M is lifting if and only if each M_i is lifting.

Proof: →) clear by [1] .

←) let N be a sub-module of M and let $\pi_j : \bigoplus_{i=1}^{n} M_i \rightarrow M_j$ be the natural projection, for all j= 1, 2... n. We claim that $N=\bigoplus_{i=1}^{n} (N\cap M_i)$, to show that let x∈

N, then x = $\sum_{i=1}^{n} x_i$, $x_i \in M_i$, clearly $\pi_i(x)=x_i$, since M is fully stable, then $x_i \in$ N for all i. Thus $x_i \in M_i \cap N$ and hence

$$N = \bigoplus_{i=1}^{n} (N \cap M_i)$$

. Now for each i, $N \cap M_i$ is a sub-module of M_i, since M_i is lifting then $N \cap M_i = A_i \oplus B_i$, where A_i is a summand of M_i and B_i is small in M_i.

$$N = \bigoplus_{i=1}^{n} (N \cap M_i) = \bigoplus_{i=1}^{n} (A_i \oplus B_i)$$

$$= (\bigoplus_{i=1}^{n} A_i) \oplus (\bigoplus_{i=1}^{n} B_i)$$

. Since A_i is a summand of M_i for all i, then one can easily show that $\bigoplus_{i=1}^{n} A_i$ is a summand of $\bigoplus_{i=1}^{n} M_i$ =M. Since B_i is small in M_i , for all i , then $\bigoplus_{i=1}^{n} B_i$ is small in $\bigoplus_{i=1}^{n} M_i$ =M. Thus M is a lifting module.

Proposition 1.9: let R be a ring. The following statements are equivalent

(1) Every free R-module is lifting.

(2) Every projective R-module is lifting.

Proof: (1)⟶(2) let M be a projective R-module. By [5], there exists an epimorphism f: F ⟶ M, where F is a free R-module. Now consider the following short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} F \xrightarrow{f} M \longrightarrow 0,$$

where i is the inclusion map. Since M is projective, then by [5] the sequence is splits and hence F ≅ kerf ⊕M. By our assumption, F is a lifting module. Since M is isomorphic to a direct summand of F, then by [1], M is a lifting module.

(2)⟶ (1) is clear.

Using a similar argument one can prove the following.

Proposition 1.10: let R be a ring. The following statements are equivalent

(1) Every finitely generated free R-module is lifting.

(2) Every finitely generated projective R-module is lifting.

lemma 1.11: Let R be a ring with J(R)=0. The following statements are equivalent:

(1) R is semisimple.

(2) Every R-module is lifting.

(3) Every projective R-module is lifting

(4) Every free R-module is lifting.

(5) Every finitely generated free R-module is lifting.

(6) Every finitely generated projective R-module is lifting.

(7) R⊕R is lifting.

Proof: (1)↔(2) by [8], and(1) → (5) clear by [5].

(5) → (1) Since R is finitely generated and free as an R-module, then R is lifting. But $J(R) = 0$, so R is semisimple.

(7) → (1) Since R is a direct summand of $R \oplus R$ is lifting, then R is lifting. But $J(R) = 0$, therefore R is semisemple. And (1) → (7) is clear.

§ 2 The purely lifting modules, basic results and examples.

In this section we introduce the concept of the pure lifting modules, and we illustrate it by examples, we also give some basic properties.

Recall that a sub-module N of an R-module M is called a pure sub-module of M if for every finitely generated ideal I (equivalently every ideal I of R), $IM \cap N = IN$, see [2].

Definition 2.1: An R-module M is called a purely lifting module if for every sub-module N of M, there exists a pure sub-module K of M such that N/K is small in M/K.

Recall that an R-module M is called pure simple if 0 and M are the only pure sub-module of M, [9].

Examples and remarks 2.2: (1) Consider the module Z as Z-module. One can easily show that Z is pure simple. Let nZ be a non-trivial sub-module of Z. If Z is purely lifting then there exists a pure sub-module K of M such that $K \subseteq nZ$ and nZ/k is small in Z/K. But Z is pure simple, so K=0 and hence nZ is small in Z which is a contradiction. Thus Z is not purely lifting.

(2) Consider the Z-module Q. It is know that Q is pure simple. One can easily show that Q is not purely lifting.

(3) Recall that an R-module M is called a hollow module if every proper sub-module of M is small in M, see [10]. Clearly that every Hollow module is purely lifting.

(4) Every lifting module is purely lifting, but the converse need not be true in general.

(5) An R-module M is said to be regular if for all $a \in M$, for all $r \in R$, there exist $x \subset R$ such that $rx\ ra = ra$ equivalently, every sub-module of M is pure [11]. Every regular R-module is purely lifting, and the converse is not true in general.

Remark 2.3: let M be a pure simple R-module, then M is purely lifting if and only if M is hollow module.

Proof:→) Let N be a proper sub-module of M. Since M is purely lifting module, then there exists a pure sub-module K of M such that K sub-module of N and N/K is small in M/K. But M is pure simple, therefore K=0 and hence N is small in M, so M is a hollow.

←) Clear.

Recall that an R-module M is called a V-module if for every factor module N of M, $Rad(N) = 0$, see [10].

Remark 2.4: Let M be a V- module, then M is purely lifting if and only if M is a regular module.

Proof: let N be a sub-module of M. Since M is purely lifting, then there exists a pure sub-module K of M such that $K \subsetneq N$ and N/K is small in M/K. But M is a V-module, then J (M/K) = 0 and hence (N/K) =0 by [5]. Thus N = K.

It is known that, if M is a regular module, then J (M) = 0, see [11]. The following example shows that a purely lifting module need not be a lifting.

Example 2.5: Let R be a regular ring which is not semisimple since R is regular, then J(R) = 0. By (1.2), there exists an R-module M such that M is not lifting. But R is a regular ring, so by [11] M is a regular module and hence M is purely lifting.

Proposition 2.6: Let N be a direct summand of a purely lifting R-module M, then N is a purely lifting.

Proof: Since N is a direct summand of M, then M = N $\oplus$ N1, for some sub-module N1 of M. Now let A be a sub-module of N. Since M is purely lifting, then there exists a pure sub-module K of M such that $K \subsetneq A$ and A/K is small in M/K. By [2] K is pure in N. Since N is a summand of M, one can easily show that N/K is a summand of M/K. But $(A/K) \subsetneq (N/K)$, so A/K is small in N/K by [4]. Thus N is purely lifting.

Recall that an R-module is said to have the pure sum property (psp) if the sum of any two pure sub-module of M is pure [12].

It's known that every commutative ring with identity has the psp [12].

Proposition 2.7: Let M be a purely lifting module such that M has the psp. if N is a pure sub–module of M ,then M/N is a purely lifting R –module .

Proof: let A/N be a sub-module of M/N. Since M is purely lifting, then there exists a pure sub–module B of M such that $B \subsetneq A$ and A/B is small in M/A. Since N is pure in M, then B+M is pure in M and hence by [5], $\frac{B+N}{N}$ is pure in M/N, clearly that $\frac{B+N}{N} \subsetneq \frac{A}{N}$ and hence $\frac{B+N}{N}$ is pure in A/N. Since A/B is small in M/B, then one can easily show that $\frac{A}{B+N}$ is small in $\frac{M}{B+N}$ .let K=B+N.
By the third isomorphism theorem: $((A/N)/(K/N)) \cong (A/K)$ and $((M/N)/(K/N)) \cong (M/K)$. So ((A/N)/(K/N)) is small in ((M/N)/ (K/N)) thus (M/N) is purely lifting.

Corollary 2.8: let R be a purely lifting ring and let I be a pure ideal of R, then R/I is purely lifting as an R –module .

Proof: Clear.

In the following theorem we will give the characterizations of a purely lifting module.

Theorem 2.9: let M be an R-module, then the following statements are equivalent

(1) M is purely lifting module.

(2) For each sub-module N of M, there exists a pure sub-module K of M such that K ⊆ N and M =K+K`, K` is a sub-module of M, N=K+S, S is a sub-module of N and (S/ (K`∩K)) is small in (K`/ (K`∩K)).

(3) For each sub-module N of M, there exists a pure sub-module K of M such that M=K+K`, for some K` sub-module of M and K⊆N and ((K`⊆N)/ (K`∩K)) is small in (K`/ (K`∩K)).

Proof: (1) → (3) Let N be a sub-module of M. Since M is purely lifting, then there exists a pure sub-module K of M such that K □N and N/K is small in M/K, clearly that M=K+M. let K`=M, then $\frac{K'\cap N}{K'\cap K}=\frac{N}{K}$ is small in $\frac{M}{K}=\frac{K'}{K\cap K'}$.

(3) → (1) let N be a sub-module of M. By our assumption, there exists a pure sub-module K of M such that K∩N, M=K+K`, K` is a sub-module of M, and $\frac{K'\cap N}{K'\cap K}$ is small in $\frac{K'}{K'\cap K}$. We claim that N/K is small in M/K. To show that let $\frac{N}{K}+\frac{B}{K}=\frac{M}{K}$, B is a sub-module of M, and hence N+B=M. Since M=K+K`, then by modular law N=K+ (K`∩N). Now M=K= (K`∩N)+B= (K`∩N)+B. By modular law K`= (K`∩N)+(K`∩B). So

$\frac{K'\cap N}{K'\cap K}+\frac{B\cap K'}{K'\cap K}=\frac{K'}{K\cap K'}$, But $\frac{K'\cap K}{K'\cap K}$ is small in $\frac{K'}{K\cap K'}$, therefore $\frac{B\cap K'}{K\cap K'}=\frac{K'}{K\cap K'}$ and hence K`∩B=K`. So K`⊆B but K⊆B, therefore K+K`=M=B.

(3) → (2) Clear, take S= (K`∩N).
(2) → (3) Let

$$\frac{K'\cap N}{K'\cap K}+\frac{B}{K'\cap K}=\frac{K'}{K\cap K'}$$

Since N=K+S and S⊆K`, then by modular law N∩K`= (K∩K`) +S, and hence

$$\frac{(K\cap K')+S}{K\cap K'}+\frac{B}{K\cap K'}=\frac{S}{K\cap K'}+\frac{B}{K\cap K'}=\frac{K'}{K\cap K'}$$

But $\frac{S}{K\cap K'}$ is small in $\frac{K'}{K\cap K'}$, therefore

$\frac{B}{K\cap K'}=\frac{K'}{K\cap K}$, thus B=K`.

§3 The direct sum of the purely lifting modules:

We start this section by the following remark

Remark 3.1: If M and N are purely lifting R-modules, then $M\oplus N$ may not be purely lifting as is seen in the following example.

Consider the ring R= $Z_8 \oplus Z_2$. One can easily show that each of the ideals $Z_8 \oplus 0$ and $0 \oplus Z_2$ is purely lifting. let N= $((\bar{2},\bar{1}))$, the only pure ideal in R contained in N is $((\bar{0},\bar{0}))$, if R is purely lifting, then N is small ideal in R, which is contradiction because N + $((\bar{1},\bar{1}))$ = R. Hence R is not purely lifting.

Proposition 3.2: Let M be a fully stable R-module such that M = $\bigoplus_{i\in I} M_i$, where each M_i is a sub-module of M, then the module M is purely lifting if and only if each M_i is purely lifting.

Proof: $\rightarrow$) Clear by (2.6).

$\leftarrow$) Let N be a sub-module of M. for each j$\in$I, let $\pi_j : M \rightarrow M_j$ be the natural projection. Each x$\in$N can be written as a finite sum x= $\sum_{x \in I} m_i$, with $m_i \in M_i$. Since N is fully invariant then $\pi_j(x) = m_j \in N \cap M_j$, so x=$\sum_{i \in I} \pi_i$ (x) . Thus

N$\subseteq \bigoplus_{i \in I} \pi_i(N) \subseteq \bigoplus_{i \in I} (N \cap M_i) \subseteq N$, or

N= $\bigoplus_{i \in I} (N \cap M_i)$ since M_i is purely lifting and $(N \cap M_i) \subseteq M_i$, then there exists a pure submodule K_i of M_i such that $K_i \subseteq (N \cap M_i)$ and $\frac{(N \cap M_i)}{K_i}$ is small in $\frac{M_i}{K_i}$, for each i$\in$I. Since K_i is pure in M_i , for each i, then by [13] $\bigoplus_{i \in I} K_i$ is pure in $\bigoplus_{i \in I} M_i$ =M. Since $\frac{(N \cap M_i)}{K_i}$ is small in $\frac{M_i}{K_i}$, for each i$\in$I, then one can easily show that

$\bigoplus_{i \in I} (\frac{N \cap M_i}{K_i})$ is small in $\bigoplus_{i \in I} (\frac{M_i}{K_i})$.

Thus $\frac{\bigoplus_{i \in I} (N \cap M_i)}{\bigoplus_{i \in I} K_i}$ is small in $\frac{\bigoplus_{i \in I} M_i}{\bigoplus_{i \in I} K_i} = \frac{M}{\bigoplus_{i \in I} K_i}$

Proposition 3.3: Let M1 and M2 be two purely lifting R-modules if ann(M1) + ann(M2) = R, then M1 $\oplus$ M2 is purely lifting module.

Proof: Let N be a sub-module of M1 $\oplus$ M2. Since ann (M1) + ann (M2) = R, then by the same way of the proof of [1, prop. (4.2), CH.1] N=N1$\oplus N2$, where N1 is a sub-module of M1 and N2 is a sub-module of M2. Since each of M1 and M2 are purely lifting, then there exists K1 pure in M1 and K2 pure in M2 such that $\frac{N1}{K1}$ is small in $\frac{M1}{K1}$, and $\frac{N2}{K2}$ is small in $\frac{M2}{K2}$.

One can easily show that K1 $\oplus K2$ is pure in M1 $\oplus M2$, and $\frac{N1 \oplus N2}{K1 \oplus K2}$ is small in $\frac{M1 \oplus M2}{K1 \oplus K2}$. Let K1 $\oplus K2$ =K. so K is pure in M1$\oplus M2$, K$\subseteq$N and $\frac{N}{K}$ is small in $\frac{M1 \oplus M2}{K}$. Thus M1 $\oplus M2$ is purely lifting.

Proposition 3.4: Let M be an R-module and let $\bar{R} = \frac{R}{annM}$. M is purely lifting as R-module if and only if M is purely lifting as $\bar{R}$-module.

Proof: Clear

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مودیلی بەرزکەرەوەی پاك.

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پوختە

لەم تویژینەوەیەدا ئیمە مۆدیلی بەرزکەرەوە دە گەشتینین سیفات و جیا کەرەوەی جیا جیا مان دروست کردوە بۆ شیوە گشتییەکە : مۆدیلی بەرزکەرەوەی پاك، وە هەندیك لە ئە نجامە زانراوە کانیشمان گەشتاندوە.

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